NORTH SYDNEY GIRLS HIGH SCHOOL

2016
TRIAL HSC EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black pen.
- Board approved calculators may be used.
- A reference sheet is provided.
- Show all necessary working in questions 11–14.

Total Marks –70

Section I Pages 3 - 6

10 Marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section.

Section II Pages 7–15

60 marks

- Attempt Questions 11–14.
- Allow about 1 hour and 45 minutes for this section.

NAME: ________________________ **STUDENT NUMBER:** ______________________________

Colour in the circle next to your teacher's name:

- ○ Mr T
- ○ Ms Narayanan
- ○ Ms Everingham
- ○ Mrs Juhn
- ○ Mr Moon
- ○ Ms Viswanathan

QUESTION	MARK
1 – 10	/10
11	/15
12	/15
13	/15
14	/15
TOTAL	/70

Section I

10 marks
Attempt Questions 1 – 10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10.

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1 Which of the following is equivalent to $\int \frac{1}{\sqrt{1-4x^2}}\,dx$?

(A) $\sin^{-1}\left(\frac{x}{2}\right)+C$

(B) $\sin^{-1}(2x)+C$

(C) $\frac{1}{2}\sin^{-1}\left(\frac{x}{2}\right)+C$

(D) $\frac{1}{2}\sin^{-1}(2x)+C$

2 The point P divides the interval from $A(3,-1)$ to $B(9,2)$ externally in the ratio 5 : 2.

What is the x coordinate of P ?

(A) $-7\frac{4}{5}$

(B) $4\frac{5}{7}$

(C) $7\frac{2}{7}$

(D) 13

3 What is the value of $\lim_{x\to 0}\frac{x\cos 3x}{\sin 2x}$?

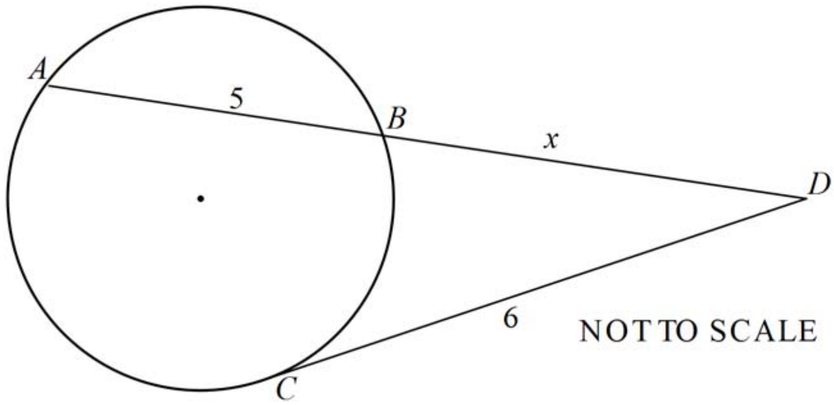
(A) $\frac{1}{2}$

(B) $\frac{2}{3}$

(C) $\frac{3}{2}$

(D) 2

4 In the diagram below, CD is a tangent to the circle at C and AB is 5 cm long. What is the length of BD in centimetres (marked with an x on the diagram)?



(A) 1

(B) $1\frac{1}{5}$

(C) $2\frac{1}{5}$

(D) 4

5 The velocity, v, of a particle in simple harmonic motion is given by $v^2 = 4 + 4x - 2x^2$.
What is the amplitude and centre of motion?

(A) amplitude is 3 and centre is $x = 1$

(B) amplitude is 3 and centre is $x = -1$

(C) amplitude is $\sqrt{3}$ and centre is $x = 1$

(D) amplitude is $\sqrt{3}$ and centre is $x = -1$

6 Which of the following pairs represents the domain and range of $y = \ln\left(1+\sqrt{4-x^2}\right)$?

(A) Domain: $-2 \le x \le 2$
Range: $y \ge 0$

(B) Domain: $-\sqrt{2} \le x \le \sqrt{2}$
Range: $y \ge \ln 3$

(C) Domain: $-2 \le x \le 2$
Range: $0 \le y \le \ln 3$

(D) Domain: $-\sqrt{2} \le x \le \sqrt{2}$
Range: All real y

7 Let α, β and γ be the roots of $P(x) = 2x^3 - 5x^2 + 4x - 9$. What is the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$?

(A) $-\frac{5}{9}$

(B) $-\frac{4}{9}$

(C) $\frac{4}{9}$

(D) $\frac{5}{9}$

8 What is the derivative of $y = \cos^{-1}\left(\frac{1}{x}\right)$?

(A) $\frac{-1}{\sqrt{x^2-1}}$

(B) $\frac{-1}{|x|\sqrt{x^2-1}}$

(C) $\frac{1}{\sqrt{x^2-1}}$

(D) $\frac{1}{|x|\sqrt{x^2-1}}$

9 Which of the following are true for all real values of x ?

I $\sin\left(\frac{\pi}{2}+x\right)=\cos\left(\frac{\pi}{2}-x\right)$

II $\sin\left(x+\frac{3\pi}{2}\right)=\cos(\pi-x)$

III $\sin x\cos x\le\frac{1}{4}$

IV $2+2\sin x-\cos^2 x\ge 0$

(A) **I** and **II**

(B) **III** and **IV**

(C) **II** and **IV**

(D) **I** and **III**

10 Oil is spilled from an oil rig in the Gulf of Mexico and spreads in a circle with the circumference changing at a rate of 40 m/s. How fast is the area of the spill increasing when the circumference of the circle is 100π m?

(A) $1500\ \mathrm{m}^2/\mathrm{s}$

(B) $2000\ \mathrm{m}^2/\mathrm{s}$

(C) $2100\ \mathrm{m}^2/\mathrm{s}$

(D) $2500\ \mathrm{m}^2/\mathrm{s}$

Section II

60 marks
Attempt Questions 11 – 14.
Allow about 1 hour and 45 minutes for this section.

Answer each question in a NEW writing booklet. Extra pages are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

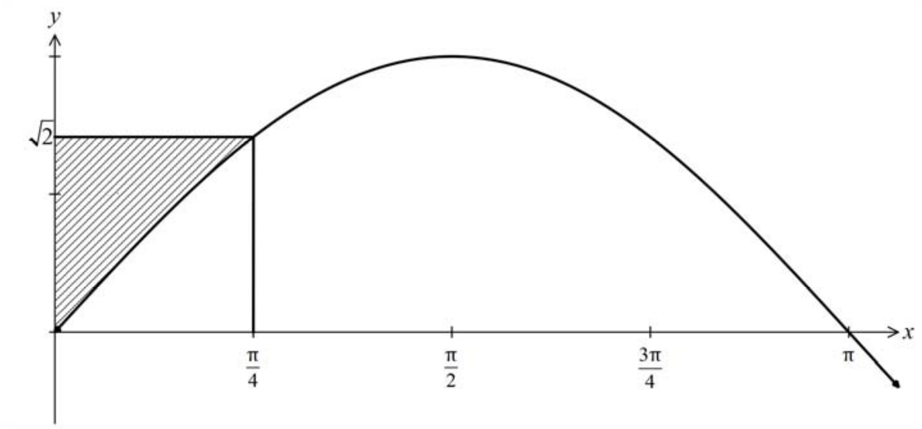
(a) Solve $2x-1 \geq \frac{6}{x}$. **3**

(b) (i) Show that $\sqrt{3}\sin x + \cos x$ can be written in the form $2\cos\left(x - \frac{\pi}{3}\right)$. **3**

(ii) Hence, or otherwise, find the value(s) of x for which $\sqrt{3}\sin x + \cos x$ is a minimum in the interval $0 \leq x \leq 2\pi$. **2**

(c) Use the substitution $u = \sqrt{x}$ to evaluate $\int_1^9 \frac{1}{x+\sqrt{x}}\,dx$. **3**

(d) The diagram below shows the shaded region bounded by the curve $y = 2\sin x$ and the y axis for $0 \leq y \leq \sqrt{2}$.



The region is rotated about the x axis to generate a solid of revolution.

(i) Show that the volume, V, of the solid is given by $V = \frac{\pi^2}{2} - \pi\int_0^{\frac{\pi}{4}} 4\sin^2 x\,dx$. **1**

(ii) Find the exact value of V. **3**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Factorise $P(x) = 2x^3 + 3x^2 - 1$, given that $x = -1$ is a zero. **2**

(ii) Solve the equation $\sqrt{x^2 + 2x + 5} = x + \sqrt{2x + 3}$. **3**

(b) Bobby has a can of cola at a temperature of $23°\text{C}$. He places the can in a fridge which has a temperature of $3°\text{C}$.

After t minutes, the temperature, c (in $°\text{C}$), of the can of cola satisfies:

$$\frac{dc}{dt} = -\frac{1}{25}(c - 3)$$

(i) Show that $c = 3 + ae^{-\frac{t}{25}}$ satisfies this equation, where a is a constant. **1**

(ii) Bobby would like to drink the can of cola when its temperature is $5°\text{C}$. **3**

If he put the can in the fridge at 8:50 a.m, when should he drink it?

Give your answer to the nearest minute.

(c) (i) Prove by mathematical induction that: **3**

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2 \text{ for integers } n \geq 1$$

(ii) Deduce that $2^3 + 4^3 + 6^3 + \ldots + (2n)^3 = 2n^2(n+1)^2$. **1**

(iii) Hence, or otherwise, find a simplified expression for: **2**

$$1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3$$

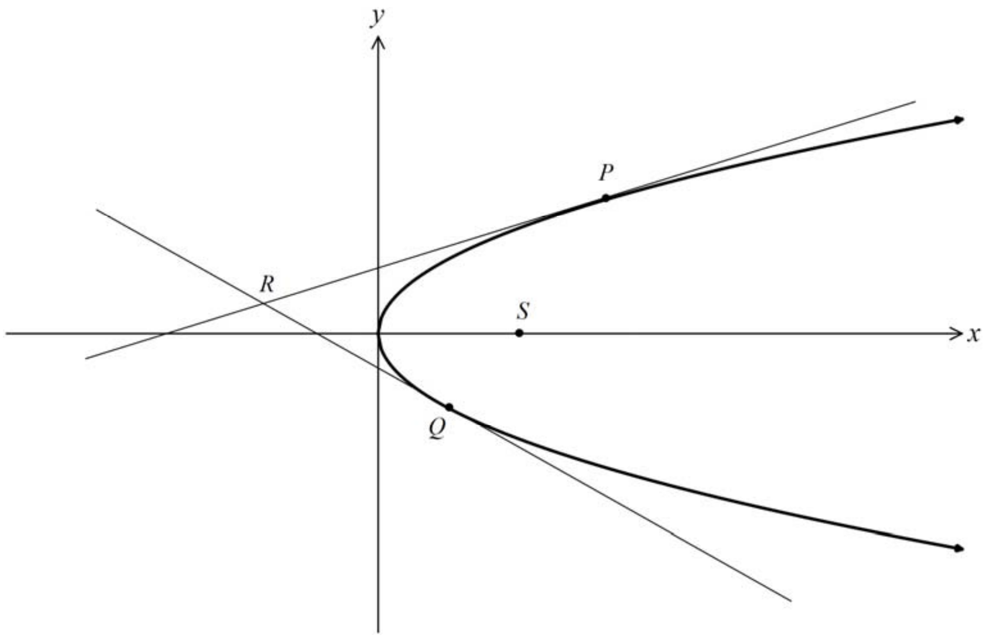
Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The point $P\left(3p^2, 6p\right)$ lies on the parabola $y^2 = 12x$ and the point S is the focus of this parabola. The point $Q\left(3q^2, 6q\right)$, where $p \neq q$, also lies on the parabola.

The tangents to the parabola at the points P and Q meet at the point R, as shown in the diagram below.

The equation of the tangent at point P is given by $y = \frac{1}{p}x + 3p$.

(Do NOT prove this).



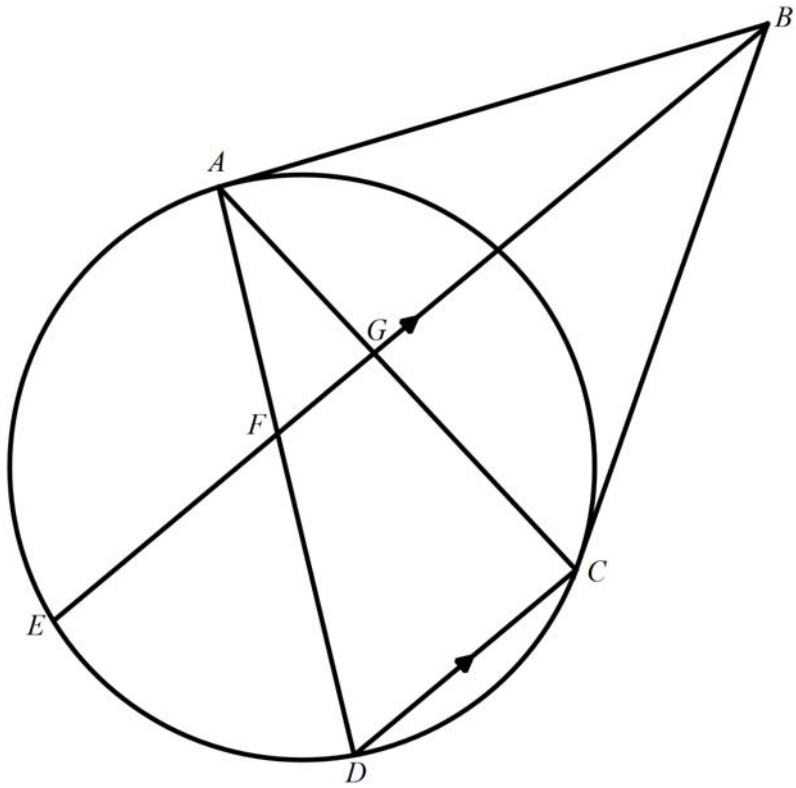
(i) Prove that $SP = 3\left(1 + p^2\right)$. **1**

(ii) Find the coordinates of R. **2**

(iii) Hence, or otherwise, prove that $SR^2 = SP \times SQ$. **2**

Question 13 continues on page 11

(b) In the diagram, *EB* is parallel to *DC*. Tangents from *B* meet the circle at *A* and *C*.



Let $\angle BCA = \alpha$.

Prove that:

(i) $\angle BCA = \angle BFA$. **2**

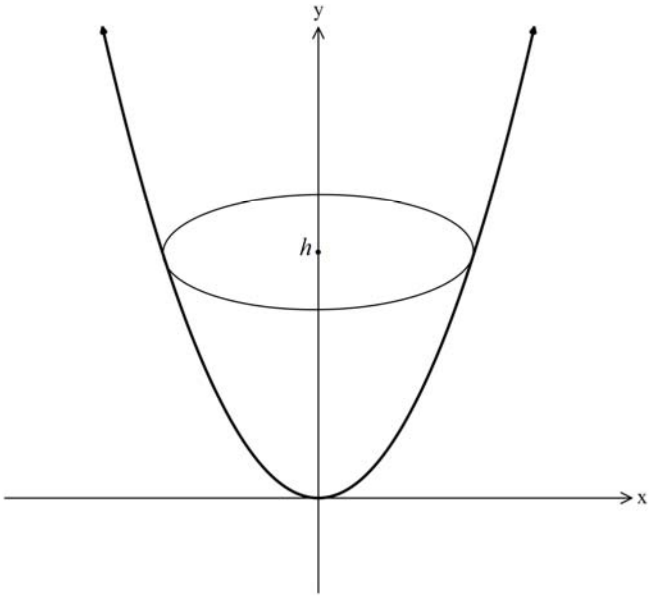
(ii) *ABCF* is a cyclic quadrilateral. **1**

(c) Find a general solution of the following equation: **2**

$$\sin\left(2x + \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$$

Question 13 continues on page 12

(d) A bowl is formed by rotating $y = x^{2n}$ (where n is an integer $n \geq 1$) about the y axis as shown.



The bowl is initially filled with water. The bowl has a small hole in its bottom (initially plugged).

When the hole is unplugged, water flows at a rate described by:

$$\frac{dV}{dt} = -\pi a^2 \sqrt{2gh}$$

where a is the radius of the hole, h is the depth of water at any time and g is a constant.

(i) Show that the volume, V, of the water in the bowl is given by: **2**

$$V = \frac{n\pi}{n+1} \cdot h^{\frac{1+n}{n}}$$

(ii) When the hole is unplugged, find the expression which describes the rate at which the water level is changing with respect to time. **2**

(iii) If the water level falls at a constant rate with respect to time, find the value of n. **1**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Consider a function $f(x)$. The domain of $y = f(x)$ is $a < x < b$ and the range is all real *y*. Assume that $f'(x)$ exists for $a < x < b$ and is non-zero.

Let $g(x)$ be the inverse function of $f(x)$. Assume that $g'(x)$ exists.

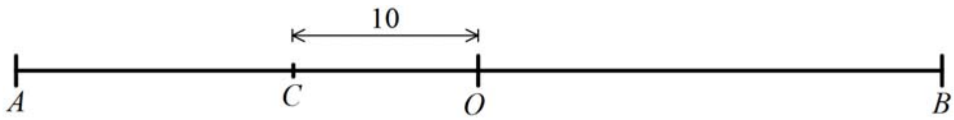
(i) Show that $g'(x) = \dfrac{1}{f'(g(x))}$. **2**

(ii) Consider $f(x) = 3 + x^2 + 2\tan\left(\dfrac{\pi x}{2}\right)$ for $-1 < x < 1$. **1**

Explain why $g(3) = 0$.

(iii) Show that the gradient of the tangent to the curve $y = g(x)$ at the point (3, 0) is $\dfrac{1}{\pi}$. **2**

(b) A particle is executing simple harmonic motion between points *A* and *B*. *O* is the centre of motion. Initially, the particle is at *C*, 10 units to the left of *O*. At this time, it is moving to the right at 6 m/s and its acceleration is 0.625 m/s^2.



The displacement of the particle after *t* seconds is given by $x = A\cos(nt + \alpha)$ for some constants *A*, *n* and α.

(i) Show that $n = \dfrac{1}{4}$. **1**

(ii) Find the values of *A* and α. **3**

(iii) *M* is the midpoint of *OB*. Find the time when the particle is first at *M*. **2**

Question 14 continues on page 15

Question 14 (continued)

(c) Car A and car B are travelling in the same direction along a straight, level road at constant speeds V_A and V_B respectively. Initially, car A is behind car B, but is travelling faster.

When car A is exactly D metres behind car B, car A applies its brakes, producing a constant acceleration $-k\ \text{m/s}^2$, where $k > 0$.

(i) Using calculus, show that the speed of car A after it has travelled D metres under braking is given by: **2**

$$v = \sqrt{{V_A}^2 - 2kD}$$

(ii) The distances travelled in t seconds by car A (after braking) and car B, are given by $x_A = V_A t - \frac{1}{2}kt^2$ and $x_B = V_B t$ respectively. **2**

(Do NOT prove this).

Prove that the cars will collide if $V_A - V_B \geq \sqrt{2kD}$.

End of paper

2016 Extension 1 Mathematics

Trial HSC Solutions

Multiple Choice

Q1	D
Q2	D
Q3	A
Q4	D
Q5	C
Q6	C
Q7	C
Q8	D
Q9	C
Q10	B

Question 1:

$$\int \frac{1}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{4\left(\frac{1}{4}-x^2\right)}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\frac{1}{4}-x^2}}\,dx$$

$$= \frac{1}{2}\sin^{-1}(2x)+C$$

Question 2:

$$x = \frac{-5(9)+2(3)}{5-2}$$

$$= \frac{-39}{3}$$

$$= 13$$

Question 3:

$$\lim_{x\to 0}\frac{x\cos 3x}{\sin 2x}=\lim_{x\to 0}\frac{x}{\sin 2x}\times\lim_{x\to 0}\left(\cos 3x\right)$$
$$=\frac{1}{2}\lim_{x\to 0}\frac{2x}{\sin 2x}\times 1$$
$$=\frac{1}{2}$$

Question 4:

$$x\left(x+5\right)=36$$
$$x^2+5x-36=0$$
$$\left(x+9\right)\left(x-4\right)=0$$
$$x=4 \quad \left(\text{as } x>0\right)$$
$$\therefore BD=4\text{ cm}$$

Question 5:

$$v^2=4+4x-2x^2$$
$$=2\left(2+2x-x^2\right)$$
$$=2\left(3-\left[x-1\right]^2\right)$$

This is in the form $v^2=n^2\left(a^2-\left[x-c\right]^2\right)$ where $a=\sqrt{3}$ and $c=1$.

OR at the ends of the motion, $v = 0$:

$$4+4x-2x^2=0$$
$$x^2-2x-2=0$$
$$\left(x-1\right)^2=3$$
$$x=1\pm\sqrt{3}$$

So centre of motion is at $x = 1$ and amplitude is $\sqrt{3}$

Question 6:

$$1+\sqrt{4-x^2}>0$$
So $\sqrt{4-x^2}>0$
$$\sqrt{\left(2-x\right)\left(2+x\right)}>0$$

Domain: $-2\le x\le 2$

As $0\le\sqrt{4-x^2}\le 2$
$$1\le 1+\sqrt{4-x^2}\le 3$$
$$\ln\left(1\right)\le\ln\left(1+\sqrt{4-x^2}\right)\le\ln\left(3\right)$$
$$0\le y\le\ln\left(3\right)$$

Range: $0\le y\le\ln\left(3\right)$

Question 7:

$$\alpha\beta+\beta\gamma+\alpha\gamma=2 \quad \text{and} \quad \alpha\beta\gamma=\frac{9}{2}$$

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$$

$$=\frac{2}{\left(\frac{9}{2}\right)}$$

$$=\frac{4}{9}$$

Question 8:

$$\frac{dy}{dx}=-\frac{1}{x^2}\times-\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}$$

$$u=\frac{1}{x}$$

$$=x^{-1}$$

$$\frac{du}{dx}=-\frac{1}{x^2}$$

$$y=\cos^{-1}u$$

$$\frac{dy}{du}=-\frac{1}{\sqrt{1-u^2}}$$

$$\frac{dy}{dx}=-\frac{1}{x^2}\times\frac{-1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}$$

$$=\frac{1}{x^2}\times\frac{\sqrt{x^2}}{\sqrt{x^2-1}}$$

$$=\frac{1}{x^2}\times\frac{|x|}{\sqrt{x^2-1}}$$

$$=\frac{1}{|x|\sqrt{x^2-1}}$$

Question 9:

A.	B.
$LHS = \sin\frac{\pi}{2}\cos x + \cos\frac{\pi}{2}\sin x$ $= \cos x$ $RHS = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$ $= \sin x$ $\neq LHS$	$LHS = \sin x\cos\frac{3\pi}{2} + \cos x\sin\frac{3\pi}{2}$ $= -\cos x$ $RHS = \cos\pi\cos x + \sin\pi\sin x$ $= -\cos x$ $= LHS$
C. $LHS = \sin x\cos x$ $= \frac{1}{2}\sin 2x$ $-\frac{1}{2} \leq \frac{1}{2}\sin 2x \leq \frac{1}{2}$ $-\frac{1}{2} \leq \sin x\cos x \leq \frac{1}{2}$	D. $LHS = 2 + 2\sin x - \cos^2 x$ $= 2 + 2\sin x - \left(1 - \sin^2 x\right)$ $= \sin^2 x + 2\sin x + 1$ $= \left(\sin x + 1\right)^2$ ≥ 0

Question 10:

$$\frac{dC}{dt} = 40 \text{ m/s}$$

$$C = 2\pi r$$

$$\frac{dC}{dr} = 2\pi$$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

When $C = 100\pi$:

$$2\pi r = 100\pi$$

$$r = 50$$

$$\frac{dC}{dt} = \frac{dC}{dr} \times \frac{dr}{dt}$$

$$40 = 2\pi \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{20}{\pi} \text{ m/s}$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$= 2\pi r \times \frac{20}{\pi}$$

$$= 40r \text{ m}^2/\text{s}$$

When $r = 50$:

$$\frac{dA}{dt} = 40 \times 50$$

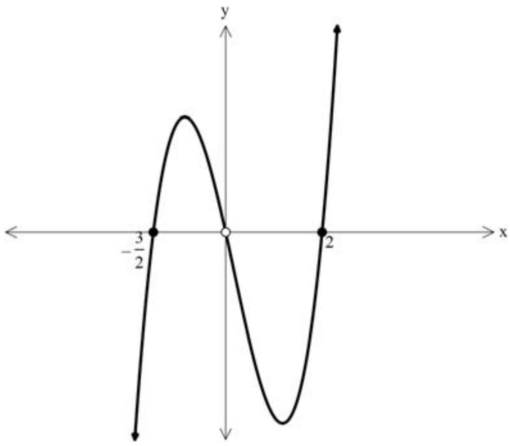
$$= 2000 \text{ m}^2/\text{s}$$

Question 11:

(a)	Solve $2x-1\geq\frac{6}{x}$.	**3**

$$x^2(2x-1)\geq 6x$$
$$2x^3-x^2-6x\geq 0$$
$$x(2x^2-x-6)\geq 0$$
$$x(2x+3)(x-2)\geq 0$$

$$-\frac{3}{2}\leq x<0,\ x\geq 2$$



(b)	(i)	Show that $\sqrt{3}\sin x+\cos x$ can be written in the form $2\cos\left(x-\frac{\pi}{3}\right)$.	**3**

$$\sqrt{3}\sin x+\cos x\equiv A\cos(x-\alpha)$$
$$\equiv A\cos x\cos\alpha+A\sin x\sin\alpha$$

Equating coefficients of cos *x* and sin *x*:

$$A\cos\alpha=1$$
$$A\sin\alpha=\sqrt{3}$$

$$A=\sqrt{1^2+\left(\sqrt{3}\right)^2}$$
$$=2$$

$$\tan\alpha=\sqrt{3}$$
$$\alpha=\frac{\pi}{3}$$

$$\therefore\sqrt{3}\sin x+\cos x\equiv 2\cos\left(x-\frac{\pi}{3}\right)$$

(ii)	Hence, or otherwise, find the value(s) of x for which $\sqrt{3}\sin x+\cos x$ is a minimum in the interval $0\le x\le 2\pi$.	**2**

$$0\le x\le 2\pi$$
$$-\frac{\pi}{3}\le x-\frac{\pi}{3}\le\frac{5\pi}{3}$$

Min. value of $2\cos\left(x-\frac{\pi}{3}\right)$ is -2.

$$2\cos\left(x-\frac{\pi}{3}\right)=-2$$
$$\cos\left(x-\frac{\pi}{3}\right)=-1$$
$$x-\frac{\pi}{3}=\pi$$
$$x=\frac{4\pi}{3}$$

(c)	Use the substitution $u=\sqrt{x}$ to evaluate $\int_1^9\frac{1}{x+\sqrt{x}}\,dx$.	**3**

$$u=\sqrt{x}$$
$$=x^{\frac{1}{2}}$$

Let $\frac{du}{dx}=\frac{1}{2}x^{-\frac{1}{2}}$

$$=\frac{1}{2\sqrt{x}}$$
$$2\sqrt{x}du=dx$$
$$dx=2udu$$

When $x=1$, $u=1$

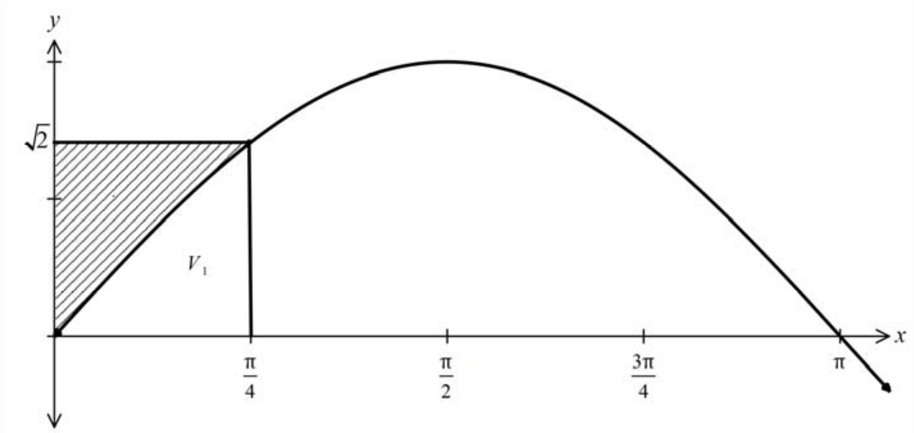
When $x=9$, $u=3$

$$\int_1^9\frac{1}{x+\sqrt{x}}dx=\int_1^3\frac{2u}{u^2+u}du$$
$$=\int_1^3\frac{2}{u+1}du$$
$$=2\Big[\ln(u+1)\Big]_1^3$$
$$=2(\ln 4-\ln 2)$$
$$=2\ln 2$$

(d) The diagram below shows the shaded region bounded by the curve $y = 2\sin x$ and the y axis for $0 \le y \le \sqrt{2}$.

The region is rotated about the x axis to generate a solid of revolution.

(i) Show that the volume, V, of the solid is given by $V = \dfrac{\pi^2}{2} - \pi\displaystyle\int_0^{\frac{\pi}{4}} 4\sin^2 x\,dx$. **1**



$$V_1 = \pi\int_0^{\frac{\pi}{4}} (2\sin x)^2\,dx$$

$$= \pi\int_0^{\frac{\pi}{4}} 4\sin^2 x\,dx$$

$$\text{Shaded volume} = \pi \times \left(\sqrt{2}\right)^2 \times \frac{\pi}{4} - \pi\int_0^{\frac{\pi}{4}} 4\sin^2 x\,dx$$

$$V = \frac{\pi^2}{2} - \pi\int_0^{\frac{\pi}{4}} 4\sin^2 x\,dx$$

(ii) Find the exact value of V. **3**

$$V = \frac{\pi^2}{2} - \pi\int_0^{\frac{\pi}{4}} 4\sin^2 x\,dx$$

$$= \frac{\pi^2}{2} - \pi\int_0^{\frac{\pi}{4}} (2 - 2\sin 2x)\,dx$$

$$= \frac{\pi^2}{2} - \pi\left[2x + \cos 2x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi^2}{2} - \pi\left\{\left(\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - \left(0 + \cos 0\right)\right\}$$

$$= \frac{\pi^2}{2} - \pi\left\{\frac{\pi}{2} - 1\right\}$$

$$= \frac{\pi^2}{2} - \frac{\pi^2}{2} + \pi$$

$$= \pi \text{ units}^3$$

Using:

$$\cos 2x = 1 - 2\sin^2 x$$
$$2\sin^2 x = 1 - \cos 2x$$
$$4\sin^2 x = 2 - 2\cos 2x$$

Question 12:

(a) (i) Factorise $P(x) = 2x^3 + 3x^2 - 1$, given that $x = -1$ is a root. **2**

$(x+1)$ is a factor of $P(x)$

$$2x^3 + 3x^2 - 1 = (x+1)(ax^2 + bx + c)$$

Equating coefficients:

$$a = 2 \text{ and } c = -1$$
$$b + a = 3$$
$$b + 2 = 3$$
$$b = 1$$

$$2x^3 + 3x^2 - 1 = (x+1)(2x^2 + x - 1)$$
$$= (x+1)(2x-1)(x+1)$$
$$= (x+1)^2(2x-1)$$

(ii) Solve the equation $\sqrt{x^2 + 2x + 5} = x + \sqrt{2x+3}$. **3**

$$\sqrt{x^2 + 2x + 5} = x + \sqrt{2x+3}$$

Square both sides:

$$\cancel{x^2} \cancel{+2x} + 5 = \cancel{x^2} + 2x\sqrt{2x+3} \cancel{+2x} + 3$$
$$2 = 2x\sqrt{2x+3}$$
$$x\sqrt{2x+3} = 1$$
$$x^2(2x+3) = 1$$
$$2x^3 + 3x - 1 = 0$$
$$(x+1)^2(2x-1) = 0$$

Test $x = \frac{1}{2}$:

$$LHS = \sqrt{\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) + 5}$$
$$= \frac{5}{2}$$
$$RHS = \frac{1}{2} + \sqrt{2\left(\frac{1}{2}\right) + 3}$$
$$= \frac{5}{2}$$
$$= LHS$$

Test $x = -1$:

$$LHS = \sqrt{(-1)^2 + 2(-1) + 5}$$
$$= 2$$
$$RHS = -1 + \sqrt{2(-1) + 3}$$
$$= 0$$
$$\neq LHS$$

The only answer is $x = \frac{1}{2}$

(b) Bobby has a can of cola at a temperature of $23°\text{C}$. He places the can in a fridge which has a temperature of $3°\text{C}$.

After *t* minutes, the temperature, *c* (in $°\text{C}$), of the can of cola satisfies:

$$\frac{dc}{dt} = -\frac{1}{25}(c-3)$$

(i) Show that $c = 3 + ae^{-\frac{t}{25}}$ satisfies this equation, where *a* is a constant. **1**

$$\frac{dc}{dt} = -\frac{1}{25}(c-3)$$

$$\begin{aligned} LHS &= \frac{dc}{dt} \\ &= \frac{d}{dt}\left(3 + ae^{-\frac{t}{25}}\right) \\ &= -\frac{a}{25}e^{-\frac{t}{25}} \end{aligned}$$

$$\begin{aligned} RHS &= -\frac{1}{25}(c-3) \\ &= -\frac{1}{25}\left(3 + ae^{-\frac{t}{25}} - 3\right) \\ &= -\frac{a}{25}e^{-\frac{t}{25}} \\ &= LHS \end{aligned}$$

So $c = 3 + ae^{-\frac{t}{25}}$ satisfies this equation.

(ii) Bobby would like to drink the can of cola when its temperature is $5°\text{C}$. **3**

If he put the can in the fridge at 8:50 a.m, when should he drink it?

Give your answer to the nearest minute.

When $t = 0$:

$$\begin{aligned} 23 &= 3 + a(1) \\ a &= 20 \end{aligned}$$

When $c = 5$:

$$\begin{aligned} 5 &= 3 + 20e^{-\frac{t}{25}} \\ 2 &= 20e^{-\frac{t}{25}} \\ \frac{1}{10} &= e^{-\frac{t}{25}} \\ e^{\frac{t}{25}} &= 10 \\ \frac{t}{25} &= \ln(10) \\ t &= 25\ln(10) \\ &= 57.5646\ldots\text{min} \end{aligned}$$

Bobby should drink the cola at 9:48 a.m.

(c)	(i) Prove by mathematical induction that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ for integers $n \geq 1$	**3**

Step 1: Prove true for $n = 1$

$$LHS = 1^3 = 1$$

$$RHS = \frac{1}{4} \times 1^2 (1+1)^2 = \frac{1}{4} \times 4 = 1 = LHS$$

Proven true for $n = 1$

Step 2: Assume true for $n = k$.

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{1}{4}k^2(k+1)^2$$

Step 3: Prove true for $n = k + 1$

RTP $1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$

$$\begin{aligned} LHS &= 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 \\ &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \text{ by assumption} \\ &= \frac{1}{4}(k+1)^2\left[k^2 + 4(k+1)\right] \\ &= \frac{1}{4}(k+1)^2\left[k^2 + 4k + 4\right] \\ &= \frac{1}{4}(k+1)^2(k+2)^2 \\ &= RHS \end{aligned}$$

By principle of mathematical induction, proven true for $n \geq 1$

(ii)	Deduce that $2^3 + 4^3 + 6^3 + \ldots + (2n)^3 = 2n^2(n+1)^2$.	**1**

$$\begin{aligned} 2^3 + 4^3 + 6^3 + \ldots + (2n)^3 &= \sum_{r=1}^{n}(2r)^3 \\ &= 8\sum_{r=1}^{n} r^3 \\ &= 8 \times \frac{1}{4}n^2(n+1)^2 \\ &= 2n^2(n+1)^2 \end{aligned}$$

(iii) Hence, or otherwise, find a simplified expression for: **2**

$$1^3+3^3+5^3+\ldots+(2n-1)^3$$

$$\begin{aligned}1^3+3^3+5^3+\ldots+(2n-1)^3 &= \left(1^3+2^3+3^3+\ldots+(2n-1)^3+(2n)^3\right)-\left(2^3+4^3+6^3+\ldots+(2n-2)^3+(2n)^3\right)\\ &=\sum_{r=1}^{2n} r^3-\sum_{r=1}^{n}(2r)^3\\ &=\frac{1}{4}(2n)^2(2n+1)^2-2n^2(n+1)^2\\ &=n^2(2n+1)^2-2n^2(n+1)^2\\ &=n^2\left[(2n+1)^2-2(n+1)^2\right]\\ &=n^2\left[4n^2+4n+1-2n^2-4n-2\right]\\ &=n^2\left(2n^2-1\right)\end{aligned}$$

Question 13:

(a) The point $P\ (3p^2, 6p)$ lies on the parabola with $y^2=12x$ and the point S is the focus of this parabola. The point $Q\ (3q^2, 6q)$, where $p\neq q$, also lies on the parabola.

The tangent to the parabola at the point P and the tangent to the parabola at Q meet at the point R, as shown in the diagram below.

The equation of the tangent at point P is given by $y=\frac{1}{p}x+3p$.

(Do NOT prove this).

(i) Prove that $SP=3(1+p^2)$ **1**

For $y^2=12x$, S is (3, 0)

$$\begin{aligned}SP&=\sqrt{(3p^2-3)^2+(6p)^2}\\ &=\sqrt{9p^4-18p^2+9+36p^2}\\ &=\sqrt{9p^4+18p^2+9}\\ &=\sqrt{9(p^4+2p^2+1)}\\ &=3\sqrt{(p^2+1)^2}\\ &=3(1+p^2)\end{aligned}$$

(ii)	Find the coordinates of *R*.	**2**

Equation tangent at *P*: $y = \frac{1}{p}x + 3p$

Equation tangent at *Q*: $y = \frac{1}{q}x + 3q$

At *R*:

$$\frac{1}{p}x + 3p = \frac{1}{q}x + 3q$$
$$\frac{1}{p}x - \frac{1}{q}x = 3q - 3p$$
$$x\left(\frac{q-p}{pq}\right) = 3(q-p)$$
$$x = 3pq$$

When $x = 3pq$:

$$y = \frac{1}{p}(3pq) + 3p$$
$$= 3q + 3p$$
$$= 3(q+p)$$

Coordinates of *R* are $(3pq, 3(p+q))$

(iii)	Hence, or otherwise, prove that $SR^2 = SP \times SQ$.	**2**

$$LHS = SR^2$$
$$= (3pq-3)^2 + (3[p+q])^2$$
$$= 9p^2q^2 - 18pq + 9 + 9(p^2 + 2pq + q^2)$$
$$= 9p^2q^2 \cancel{-18pq} + 9 + 9p^2 \cancel{+18pq} + 9q^2$$
$$= 9(p^2q^2 + p^2 + q^2 + 1)$$
$$= 9(p^2[q^2+1] + 1[q^2+1])$$
$$= 9(p^2+1)(q^2+1)$$

$$RHS = SP \times SQ$$
$$= 3(1+p^2) \times 3(1+q^2)$$
$$= 9(p^2+1)(q^2+1)$$
$$= LHS$$

$$\therefore SR^2 = SP \times SQ$$

(b) In the diagram, EB is parallel to DC. Tangents from B meet the circle at A and C.

Let $\angle BCA = \alpha$

Prove that:

(i) $\angle BCA = \angle BFA$. **2**

$$\angle CDA = \angle BCA \left(\text{angle between tangent and chord is equal to angle in the alternate segment}\right)$$
$$= \alpha$$
$$\angle BFA = \angle CDA \left(\text{corresponding angles; } EB \parallel DC\right)$$
$$= \alpha$$
$$\therefore \angle BCA = \angle BFA$$

(ii) $ABCF$ is a cyclic quadrilateral. **1**

$\angle BCA = \angle BFA$ (proven)

$\therefore ABCF$ is a cyclic quadrilateral (angles in the same segment)

(c) Find a general solution of the following equation: **2**

$$\sin\left(2x + \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$$

$$\sin\left(2x + \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$$
$$\left(2x + \frac{\pi}{4}\right) = n\pi + (-1)^n \frac{\pi}{3}$$
$$2x = n\pi + (-1)^n \frac{\pi}{3} - \frac{\pi}{4}$$
$$x = \frac{n\pi}{2} - \frac{\pi}{8} + (-1)^n \frac{\pi}{6}$$
$$= \frac{4n\pi - \pi}{8} + (-1)^n \frac{\pi}{6}$$
$$= \frac{\pi(4n-1)}{8} + (-1)^n \frac{\pi}{6}$$

(d) A bowl is formed by rotating $y = x^{2n}$ (where n is an integer $n \geq 1$) about the y axis as shown.

The bowl is initially filled with water to a depth h. The bowl has a small hole in its bottom (initially plugged).

When the hole is unplugged, water flows at a rate described by:

$$\frac{dV}{dt} = -\pi a^2 \sqrt{2gh}$$

where a is the radius of the hole and g is a constant.

(i) Show that the volume, V, of the water in the bowl is given by: **2**

$$V = \frac{n\pi}{n+1} \cdot h^{\frac{1+n}{n}}$$

where h is the depth of the water in the bowl.

$$y = x^{2n}$$

$$\left(x^{2n}\right)^{\frac{1}{2n}} = y^{\frac{1}{2n}}$$

$$x = y^{\frac{1}{2n}}$$

$$V = \pi \int_0^h x^2 dy$$

$$= \pi \int_0^h \left(y^{\frac{1}{2n}}\right)^2 dy$$

$$= \pi \int_0^h y^{\frac{1}{n}} dy$$

$$= \pi \left[\frac{y^{\frac{1}{n}+1}}{\frac{1}{n}+1} \right]_0^h$$

$$= \pi \left\{ \frac{h^{\frac{1}{n}+1}}{\frac{1}{n}+1} - 0 \right\}$$

$$= \pi \left\{ \frac{h^{\frac{1+n}{n}}}{\frac{1+n}{n}} \right\}$$

$$= \frac{n\pi}{1+n} \cdot h^{\frac{1+n}{n}}$$

(ii)	When the hole is unplugged, find the expression which describes the rate at which the water level is changing with respect to time.		**2**

$$V = \frac{n\pi}{n+1} \cdot h^{\frac{1+n}{n}}$$
$$= \frac{n\pi}{n+1} \cdot h^{\frac{1}{n}+1}$$
$$\frac{dV}{dh} = \frac{n\pi}{n+1} \cdot \left(\frac{1}{n}+1\right) h^{\frac{1}{n}}$$
$$= \frac{n\pi}{n+1}\left(\frac{1+n}{n}\right) h^{\frac{1}{n}}$$
$$= \pi h^{\frac{1}{n}}$$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$
$$-\pi a^2 \sqrt{2gh} = \pi h^{\frac{1}{n}} \times \frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{-a^2\sqrt{2gh}}{h^{\frac{1}{n}}}$$

(iii)	If the water level falls at a constant rate with respect to time, find the value of n.		**1**

$$\frac{dh}{dt} = \frac{-a^2\sqrt{2g}h^{\frac{1}{2}}}{h^{\frac{1}{n}}}$$
$$= -a^2 \cdot \sqrt{2g} \cdot h^{\frac{1}{2}-\frac{1}{n}}$$

For $\frac{dh}{dt}$ to be a constant:

$$\frac{1}{2} - \frac{1}{n} = 0$$
$$n = 2$$

Question 14:

(a) Consider a function $f(x)$. The domain of $y = f(x)$ is $a < x < b$ and the range is all real *y*. Assume that $f'(x)$ exists for $a < x < b$ and is non-zero.

Let $g(x)$ be the inverse function of $f(x)$. Assume that $g'(x)$ exists.

(i) Show that $g'(x) = \dfrac{1}{f'(g(x))}$. **2**

Using $f(g(x)) = x$

Differentiate both sides:

$$f'(g(x))g'(x) = 1$$

$$g'(x) = \frac{1}{f'(g(x))}$$

(ii) Consider $f(x) = 3 + x^2 + 2\tan\left(\dfrac{\pi x}{2}\right)$ for $-1 < x < 1$. **1**

Explain why $g(3) = 0$.

$$\begin{aligned} f(0) &= 3 + 0^2 + 2\tan 0 \\ &= 3 \end{aligned}$$

Since $g(x)$ is the inverse of $f(x)$, $g(3) = 0$

(iii) Show that the gradient of the tangent to the curve $y = g(x)$ at the point (3, 0) is $\dfrac{1}{\pi}$. **2**

$$g'(x) = \frac{1}{f'(g(x))}$$

$$\begin{aligned} g'(3) &= \frac{1}{f'(g(3))} \\ &= \frac{1}{f'(0)} \end{aligned}$$

$$f(x) = 3 + x^2 + 2\tan\left(\frac{\pi x}{2}\right)$$

$$f'(x) = 2x + 2\cdot\frac{\pi}{2}\sec^2\left(\frac{\pi x}{2}\right)$$

$$= 2x + \pi\sec^2\left(\frac{\pi x}{2}\right)$$

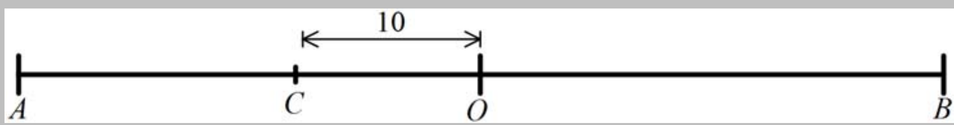
$$f'(0) = 2(0) + \pi\sec^2 0$$

$$= \pi$$

So $g'(3) = \frac{1}{\pi}$

and the gradient of the tangent at $x = 3$ is $\frac{1}{\pi}$.

(b) A particle is executing simple harmonic motion between points *A* and *B*.
O is the centre of motion. Initially, the particle is at *C*, 10 units to the left of *O*.
At this time, it is moving to the right at 6 m/s and its acceleration is 0.625 m/s^2.



The displacement of the particle after *t* seconds is given by $x = A\cos(nt + \alpha)$ for some constants *A*, *n* and α.

(i) Show that $n = \frac{1}{4}$. **1**

$$x = A\cos(nt + \alpha)$$

$$\dot{x} = -An\sin(nt + \alpha)$$

$$\ddot{x} = -An^2\cos(nt + \alpha)$$

$$= -n^2 \times A\cos(nt + \alpha)$$

$$= -n^2 x$$

When $x = -10$, $\ddot{x} = 0.625$:

$$0.625 = -n^2 \times -10$$

$$n^2 = 0.0625$$

$$n = 0.25$$

(ii)	Find the values of A and α.	3

$$x = A\cos\left(\frac{1}{4}t+\alpha\right)$$
$$\dot{x} = -\frac{A}{4}\sin\left(\frac{1}{4}t+\alpha\right)$$

When $t = 0$:

$$-10 = A\cos(\alpha) --- (1)$$
$$6 = -\frac{A}{4}\sin(\alpha)$$
$$-24 = A\sin(\alpha) --- (2)$$

To find A:

$$A = \sqrt{(-10)^2+(-24)^2}$$
$$= 26$$

As both $\sin\alpha$ and $\cos\alpha$ are negative, α is in 3rd quadrant.

To find α (sub into 1):

$$-10 = 13\cos\alpha$$
$$\cos\alpha = -\frac{5}{13}$$
$$\alpha \approx \pi + \cos^{-1}\left(\frac{5}{13}\right)$$
$$\approx 4.3176$$

(iii)	M is the midpoint of OB. Find the time when the particle is first at M.	2

As the amplitude is 26, at M, $x = 13$:

$$13 = 26\cos\left(\frac{1}{4}t+4.3176...\right)$$
$$\frac{1}{2} = \cos\left(\frac{1}{4}t+4.3176...\right)$$
$$\left(\frac{1}{4}t+4.3176...\right) = \frac{\pi}{3}, \frac{5\pi}{3}, \ldots$$
$$\left(\frac{1}{4}t+4.3176...\right) = \frac{5\pi}{3} \quad (\text{as } t \geq 0)$$
$$t \approx 3.67\text{s}$$

(c) Car *A* and car *B* are travelling in the same direction along a straight, level road at constant speeds V_A and V_B respectively. Initially, car *A* is behind car *B*, but is travelling faster.

When car *A* is exactly *D* metres behind car *B*, car *A* applies its brakes, producing a constant acceleration $-k\ \text{m/s}^2$, where $k > 0$.

(i) Using calculus, show that the speed of car *A* after it has travelled *D* metres under braking is given by: **2**

$$v = \sqrt{V_A^{\ 2} - 2kD}$$

$$\ddot{x} = -k$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -k$$

$$\frac{1}{2}v^2 = -kx + C$$

When $x = 0$, $v = v_A$:

$$\frac{1}{2}v_A^{\ 2} = C$$

$$\frac{1}{2}v^2 = -kx + \frac{1}{2}v_A^{\ 2}$$

$$v^2 = -2kx + v_A^{\ 2}$$

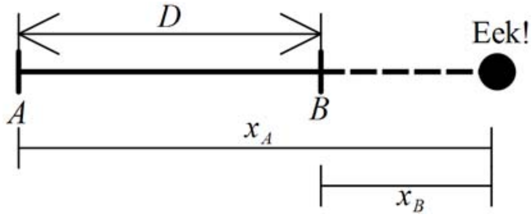
When $x = D$:

$$v^2 = -2kD + v_A^{\ 2}$$

$$v = \sqrt{v_A^{\ 2} - 2kD} \quad \text{(since speed cannot be negative)}$$

(ii)	The distances travelled in t seconds by car A (after braking) and car B, are given by $x_A = V_A t - \frac{1}{2}kt^2$ and $x_B = V_B t$ respectively. (Do NOT prove this). Prove that the cars will collide if $V_A - V_B \geq \sqrt{2kD}$.		**2**

When car A is exactly D m behind B, upon braking:



$x_B = x_A - D$:

$$V_B t = V_A t - \frac{1}{2}kt^2 - D$$

$$-\frac{1}{2}kt^2 + (V_A - V_B)t - D = 0$$

$$t = \frac{(V_B - V_A) \pm \sqrt{(V_A - V_B)^2 - 4 \cdot \left(-\frac{1}{2}k\right)(-D)}}{2\left(-\frac{1}{2}k\right)}$$

$$= \frac{(V_B - V_A) \pm \sqrt{(V_A - V_B)^2 - 2kD}}{-k}$$

Real roots occur when $\Delta \geq 0$:

$$\Delta = (V_A - V_B)^2 - 2kD$$

$$(V_A - V_B)^2 - 2kD \geq 0$$

$$(V_A - V_B)^2 \geq 2kD$$

$$(V_A - V_B) \geq \sqrt{2kD} \quad (\text{as speed is positive})$$